The binary black-hole dynamics at the third post-Newtonian order in the orbital motion

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The orbital dynamics of the binary point-mass systems is ambiguous at the third post-Newtonian order of approximation. The static ambiguity is known to be related to the difference between the Brill-Lindquist solution and the Misner-Lindquist solution of the time-symmetric conformally flat initial value problem for binary black holes. The kinetic ambiguity is noticed to violate in general the standard relation between the center-of-mass velocity and the total linear momentum as demanded by global Lorentz invariance.

I. INTRODUCTION

The motion of binary systems is one of the most important problems in general relativity, particularly in view of the future detection of gravitational waves from such systems. The simplest two-body problem is the one where the components of the system are objects which are as spherically symmetric and point-like as possible. In general relativity, those objects are black holes. In spite of the extension of black holes, in binary orbit approximate calculations of post-Newtonian (PN) type (which use expansions in powers of $1/c$, where c denotes the velocity of light), the ansatz of Dirac delta functions to describe point-like sources turned out to be very successful up to the 2.5PN approximation [1,2], and at the 3.5PN order of approximation too [3]. At the 3PN order, i.e. at the order $(1/c^2)^3$, only partial success was achievable [4]. Two terms in the binary point-mass calculations came out ambiguously: a kinetic one, depending on the bodies' linear momenta squared [4], and a static one, not depending on the momenta [5]. The authors were able to show that the static ambiguity is intimately related to the fact that different exact solutions (Brill-Lindquist and Misner-Lindquist) of the time-symmetric and conformally flat initial value problem for two black holes do exist, which at the time reveal the emergence of black holes in binary point-mass calculations at the 3PN order [5]. In this paper we confront the kinetic ambiguity with the center-of-mass motion and claim that it can be fixed through the standard relation between the center-of-mass velocity and the total linear momentum which results from global Lorentz invariance.

We employ the following notation: $\mathbf{x}_a = \left(x_a^i\right)$ $(a = 1, 2;\ i = 1, 2, 3)$ denotes the position of the ath point mass in the 3-dimensional Euclidean space endowed with a standard Euclidean scalar product (denoted by a dot). We also define $\mathbf{r}_{12} \equiv \mathbf{x}_1 - \mathbf{x}_2$, $r_{12} \equiv |\mathbf{r}_{12}|$, $\mathbf{n}_{12} \equiv \mathbf{r}_{12}/r_{12}$; $|\cdot|$ stands here for the Euclidean length of a vector. The linear momentum of the ath body is denoted by $\mathbf{p}_a = (p_{ai})$.

II. THE GENERALIZED CONSERVATIVE HAMILTONIAN TO 3PN ORDER

The post-Newtonian approximation order 2.5PN (as well 3.5PN) is of purely dissipative character. Subtraction of the 2.5PN terms from the 3PN Hamiltonian results in the conservative 3PN Hamiltonian. This Hamiltonian is primarily of higher order as it depends on time derivatives of the positions and momenta of the bodies [4].

The conservative 3PN higher-order Hamiltonian for two-body point-mass systems was calculated in Ref. [4]. We present here the explicit form of this Hamiltonian in the center-of-mass reference frame (where $\mathbf{p}_1 + \mathbf{p}_2 = 0$). It is convenient to introduce the following reduced variables

$$\mathbf{r} \equiv \frac{\mathbf{x}_1 - \mathbf{x}_2}{GM}, \quad \mathbf{p} \equiv \frac{\mathbf{p}_1}{\mu} = -\frac{\mathbf{p}_2}{\mu}, \quad \widehat{t} \equiv \frac{t}{GM}, \quad \widehat{H}^{\mathrm{NR}} \equiv \frac{H^{\mathrm{NR}}}{\mu}, \tag{1}$$

where

$$\mu \equiv \frac{m_1 m_2}{M}, \quad M \equiv m_1 + m_2, \quad \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \tag{2}$$

In Eq. (1) the superscript NR denotes a 'non-relativistic' Hamiltonian, i.e. the Hamiltonian without the rest-mass contribution Mc^2. We also introduce the reduced-time derivatives:

$$\dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{d\widehat{t}} = \frac{d(\mathbf{x}_1 - \mathbf{x}_2)}{dt}, \quad \dot{\mathbf{p}} \equiv \frac{d\mathbf{p}}{d\widehat{t}} = \frac{G}{\nu}\frac{d\mathbf{p}_1}{dt} = -\frac{G}{\nu}\frac{d\mathbf{p}_2}{dt}. \tag{3}$$

The reduced two-point-mass conservative Hamiltonian up to the 3PN order reads

$$\begin{aligned}\widehat{H}^{\mathrm{NR}}(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}}) &= \widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) + \frac{1}{c^2}\widehat{H}_{\mathrm{1PN}}(\mathbf{r},\mathbf{p}) \\ &\quad + \frac{1}{c^4}\widehat{H}_{\mathrm{2PN}}(\mathbf{r},\mathbf{p}) + \frac{1}{c^6}\widehat{H}_{\mathrm{3PN}}(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}}),\end{aligned} \tag{4}$$

where (here $r \equiv |\mathbf{r}|$ and $\mathbf{n} \equiv \mathbf{r}/r$)

$$\widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \tag{5}$$

$$\widehat{H}_{\mathrm{1PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left[(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{r} + \frac{1}{2r^2}, \tag{6}$$

$$\begin{aligned}\widehat{H}_{\mathrm{2PN}}(\mathbf{r},\mathbf{p}) &= \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 \\ &\quad + \frac{1}{8}\left[\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\right]\frac{1}{r} \\ &\quad + \frac{1}{2}\left[(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\right]\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},\end{aligned} \tag{7}$$

$$\begin{aligned}\widehat{H}_{\mathrm{3PN}}(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}}) &= \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 \\ &\quad + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-6\nu^3\right)(\mathbf{p}^2)^3 \\ &\quad + (1-2\nu)\nu^2\left[2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2\right]\Big\}\frac{1}{r} \\ &\quad + \frac{1}{48}\Big[3\left(-27+140\nu+96\nu^2\right)(\mathbf{p}^2)^2 \\ &\quad + 6(8+25\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - (35-267\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big]\frac{1}{r^2} \\ &\quad + \frac{1}{1536}\Big\{\left[-4800-3(8944-315\pi^2)\nu-7808\nu^2\right]\mathbf{p}^2 \\ &\quad + 9\left(2672-315\pi^2+448\nu\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^3} \\ &\quad + \frac{1}{96}\Big[12+(872-63\pi^2)\nu\Big]\frac{1}{r^4} \\ &\quad + \widehat{D}(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}}) + \widehat{\Omega}(\mathbf{r},\mathbf{p}).\end{aligned} \tag{8}$$

In Eq. (8) $\widehat{D}(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}})$ denotes that part of the 3PN Hamiltonian $\widehat{H}_{\mathrm{3PN}}$ which depends on the time derivatives $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$. Its explicit form reads

$$\widehat{D} = \widehat{D}_1\, r + \widehat{D}_0 + \widehat{D}_{-1}\,\frac{1}{r} + \widehat{D}_{-2}\,\frac{1}{r^2} + \widehat{D}_{-3}\,\frac{1}{r^3}, \tag{9}$$

where

$$\widehat{D}_1 = \frac{1}{12}\nu^2\Big[4(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{p}})(\mathbf{p}\cdot\dot{\mathbf{p}}) - 5(\mathbf{n}\cdot\dot{\mathbf{p}})^2\mathbf{p}^2$$
$$-5(\mathbf{n}\cdot\mathbf{p})^2\dot{\mathbf{p}}^2 - (\mathbf{n}\cdot\mathbf{p})^2(\mathbf{n}\cdot\dot{\mathbf{p}})^2 + 13\mathbf{p}^2\dot{\mathbf{p}}^2 + 2(\mathbf{p}\cdot\dot{\mathbf{p}})^2\Big], \tag{10}$$

$$\widehat{D}_0 = \frac{1}{8}\nu^2\Big\{(\mathbf{n}\cdot\dot{\mathbf{r}})(\mathbf{p}\cdot\dot{\mathbf{p}})\big[5\mathbf{p}^2 + (\mathbf{n}\cdot\mathbf{p})^2\big]$$
$$-(\mathbf{n}\cdot\mathbf{p})\big[\mathbf{p}^2(\dot{\mathbf{p}}\cdot\dot{\mathbf{r}}) + 2(\mathbf{p}\cdot\dot{\mathbf{p}})(\mathbf{p}\cdot\dot{\mathbf{r}})\big] - \frac{1}{3}(\mathbf{n}\cdot\mathbf{p})^3(\dot{\mathbf{p}}\cdot\dot{\mathbf{r}})$$
$$+(\mathbf{n}\cdot\dot{\mathbf{p}})\big[\mathbf{p}^2 + (\mathbf{n}\cdot\mathbf{p})^2\big]\big[(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{r}}) - (\mathbf{p}\cdot\dot{\mathbf{r}})\big]\Big\}, \tag{11}$$

$$\widehat{D}_{-1} = \frac{1}{24}\nu\Big\{2(17-10\nu)(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{p}})(\mathbf{n}\cdot\dot{\mathbf{r}}) - (15+22\nu)(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{n}\cdot\dot{\mathbf{p}})$$
$$-(51-8\nu)(\mathbf{n}\cdot\dot{\mathbf{p}})\mathbf{p}^2 - 2(6-5\nu)(\mathbf{n}\cdot\mathbf{p})(\mathbf{p}\cdot\dot{\mathbf{p}})$$
$$-2(1-2\nu)(\mathbf{n}\cdot\dot{\mathbf{r}})(\mathbf{p}\cdot\dot{\mathbf{p}}) - 2(7-2\nu)\big[(\mathbf{n}\cdot\mathbf{p})(\dot{\mathbf{p}}\cdot\dot{\mathbf{r}}) + (\mathbf{n}\cdot\dot{\mathbf{p}})(\mathbf{p}\cdot\dot{\mathbf{r}})\big]\Big\}$$
$$+\frac{1}{16}\nu^3\Big[8(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{r}})\mathbf{p}^2(\mathbf{p}\cdot\dot{\mathbf{r}}) + 8(\mathbf{n}\cdot\mathbf{p})^3(\mathbf{n}\cdot\dot{\mathbf{r}})(\mathbf{p}\cdot\dot{\mathbf{r}})$$
$$+2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2\dot{\mathbf{r}}^2 + 5(\mathbf{p}^2)^2\dot{\mathbf{r}}^2 + (\mathbf{n}\cdot\mathbf{p})^4\dot{\mathbf{r}}^2 - 5(\mathbf{n}\cdot\dot{\mathbf{r}})^2(\mathbf{p}^2)^2$$
$$-6(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{n}\cdot\dot{\mathbf{r}})^2\mathbf{p}^2 - 5(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{n}\cdot\dot{\mathbf{r}})^2$$
$$-4(\mathbf{n}\cdot\mathbf{p})^2(\dot{\mathbf{p}}\cdot\dot{\mathbf{r}})^2 - 4\mathbf{p}^2(\dot{\mathbf{p}}\cdot\dot{\mathbf{r}})^2\Big], \tag{12}$$

$$\widehat{D}_{-2} = \frac{1}{48}\nu\Big[5(5-7\nu)(\mathbf{n}\cdot\mathbf{p})^3(\mathbf{n}\cdot\dot{\mathbf{r}}) + 10(3-5\nu)(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{n}\cdot\dot{\mathbf{r}})^2$$
$$+3(17-35\nu)(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{r}})\mathbf{p}^2 - 28(3-8\nu)(\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{r}})(\mathbf{p}\cdot\dot{\mathbf{r}})$$
$$-15(2-3\nu)(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}\cdot\dot{\mathbf{r}}) + 2(24-77\nu)(\mathbf{n}\cdot\mathbf{p})^2\dot{\mathbf{r}}^2$$
$$+2(9-29\nu)(\mathbf{n}\cdot\dot{\mathbf{r}})^2\mathbf{p}^2 - 3(4-9\nu)\mathbf{p}^2(\mathbf{p}\cdot\dot{\mathbf{r}})$$
$$-2(12-37\nu)\mathbf{p}^2\dot{\mathbf{r}}^2 + 4(6-17\nu)(\mathbf{p}\cdot\dot{\mathbf{r}})^2\Big], \tag{13}$$

$$\widehat{D}_{-3} = \frac{1}{1536}\nu\Big\{3\big[927\pi^2 - 10832 + 36\left(48-5\pi^2\right)\nu\big](\mathbf{n}\cdot\mathbf{p})(\mathbf{n}\cdot\dot{\mathbf{r}})$$
$$-6\big[3(16+\pi^2) - 2(45\pi^2-464)\nu\big](\mathbf{n}\cdot\dot{\mathbf{r}})^2$$
$$+\big[11600 - 927\pi^2 + 20(9\pi^2-80)\nu\big](\mathbf{p}\cdot\dot{\mathbf{r}})$$
$$+2\big[176 + 3\pi^2 + 6(176-15\pi^2)\nu\big]\dot{\mathbf{r}}^2\Big\}. \tag{14}$$

The last term in Eq. (8), $\widehat{\Omega}\,(\mathbf{r},\mathbf{p})$, contains the parameters which parametrize the ambiguities of the 3PN Hamiltonian. Its explicit form is

$$\widehat{\Omega}\,(\mathbf{r},\mathbf{p}) = \omega_{\text{kinetic}}\big[\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\big]\frac{\nu^2}{r^3} + \omega_{\text{static}}\frac{\nu}{r^4}, \tag{15}$$

where ω_{kinetic} parametrizes the kinetic ambiguity, and the static ambiguity is parametrized by ω_{static}.

The 3PN higher-order Hamiltonian $\widehat{H}_{\text{3PN}}\,(\mathbf{r},\mathbf{p},\dot{\mathbf{r}},\dot{\mathbf{p}})$, Eq. (8), can be reduced to a usual Hamiltonian depending only on $\mathbf{r}$ and $\mathbf{p}$. Details of the reduction can be found in Sec. II of Ref. [6].

III. THE AMBIGUOUS PART OF THE 3PN HAMILTONIAN

It was found in [4], [5] that the ambiguous part Ω of the 3PN Hamiltonian, Eq. (15), takes, in a non-center-of-mass reference frame and in non-reduced variables, the form (G, the Newtonian gravitational constant, is put equal to one)

$$\Omega(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{c^6}\sum_a\sum_{b\neq a}\Bigg\{\omega_{\text{static}}\frac{m_a^3 m_b^2}{r_{ab}^4} + \frac{1}{2}\omega_{\text{kinetic}}\frac{m_a\, m_b}{r_{ab}^3}\left[(\mathbf{p}_a)^2 - 3(\mathbf{n}_{ab}\cdot\mathbf{p}_a)^2\right]\Bigg\}. \tag{16}$$

In the following we shall show how center-of-mass considerations can impose restrictions onto the "kinetic" ambiguous term. To make contact with the intended center-of-mass motion calculations we transform (16) to the Lagrangean level. On this level, Ω appears as $-\Omega$ with

$$\Omega(\mathbf{x}_1,\mathbf{x}_2,\mathbf{v}_1,\mathbf{v}_2) = \frac{1}{c^6}\sum_a\sum_{b\neq a}\Bigg\{\omega_{\text{static}}\frac{m_a^3 m_b^2}{r_{ab}^4} + \frac{1}{2}\omega_{\text{kinetic}}\frac{m_a^3\, m_b}{r_{ab}^3}\left[(\mathbf{v}_a)^2 - 3(\mathbf{n}_{ab}\mathbf{v}_a)^2\right]\Bigg\}, \tag{17}$$

where the bodies' velocities are given by $\mathbf{v}_a = \mathbf{p}_a/m_a$.

In view of the relation between the total linear momentum $\mathbf{P}$ and the center-of-mass velocity $\mathbf{V} = \dot{\mathbf{X}}$ ($\mathbf{X}$ denotes the center-of-mass coordinate) demanded by global Lorentz invariance, [7],

$$\mathbf{P} = \frac{H}{c^2}\frac{d\mathbf{X}}{dt}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \tag{18}$$

where H is the conserved total energy, $H = Mc^2 + H^{\text{NR}}$, the following expression (which is one of the contributions to $\mathbf{P} = \partial L/\partial \mathbf{v}_1 + \partial L/\partial \mathbf{v}_2$)

$$\mathbf{D} \equiv -\frac{\partial\Omega}{\partial\mathbf{v}_1} - \frac{\partial\Omega}{\partial\mathbf{v}_2} = -\omega_{\text{kinetic}}\frac{1}{c^6}\sum_a\sum_{b\neq a}\frac{m_a^3\, m_b}{r_{ab}^3}\left[\mathbf{v}_a - 3(\mathbf{n}_{ab}\mathbf{v}_a)\mathbf{n}_{ab}\right] \tag{19}$$

has either to be a total time derivative (in which case no restrictions are imposed on Ω) or to be fixed by other terms in Eq. (18) (note that no static ambiguity is involved here).

The simplest way to show that $\mathbf{D}$ is not a total time derivative is to write it in the form

$$D^i = \omega_{\text{kinetic}} m_1 m_2[(m_1^2 + m_2^2)V^j + \frac{m_1 m_2(m_1 - m_2)}{M}v^j]\partial_{1j}\partial_{1i}\frac{1}{r_{12}}, \tag{20}$$

(v^i denotes the relative velocity, $v^i = v_1^i - v_2^i$, and here, $MV^i = m_1 v_1^i + m_2 v_2^i$ holds) and to average D^i over a circular orbit of the relative motion simply assuming that the center-of-mass motion is orthogonal to the relative motion. The result is (notice that the relative motion part in the Eq. (20) is a total time derivative)

$$<\mathbf{D}> = -\omega_{\text{kinetic}}\frac{m_1 m_2}{r_{12}^3}(m_1^2 + m_2^2)\mathbf{V}. \tag{21}$$

In the center-of-mass frame D^i is always a total time derivative. This fits with Ref. [8] where the kinetic ambiguity, in the center-of-mass frame, was shown to be influenced by coordinate transformations which additionally only influence, unimportant in this context, the static potential at 3PN order.

IV. CONCLUSIONS

In this paper we have shown that the kinetic ambiguity obtained in binary point-mass calculations at the 3PN order should be getting fixed by center-of-mass motion considerations. The static ambiguity was known to be fixable by referring to e.g., the Brill-Lindquist ($\omega_{\rm static} = 0$) or the Misner-Lindquist solution ($\omega_{\rm static} = -1/8$). The energy content of the latter solution is influenced by the topology of the involved non-simply connected 3-space.

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